

Renormalization scheme and higher loop stability in hadronic τ decay within analytic perturbation theory

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Received: 17 September 1999 / Revised version: 13 January 2000 /
Published online: 6 April 2000 – © Springer-Verlag 2000

Abstract. We apply an analytic description to the inclusive decay of the τ lepton. We argue that this method gives not only a self-consistent description of the process both in the timelike region by using the initial expression for R_τ and in the spacelike domain by using the analytic properties of the hadronic correlator, but also leads to the fact that theoretical uncertainties associated with unknown higher-loop contributions and renormalization scheme dependence can be reduced dramatically.

1 Introduction

The ratio of hadronic to leptonic widths for the inclusive decay of the τ -lepton, $R_\tau = \Gamma(\tau^- \rightarrow \text{hadrons } \nu_\tau) / \Gamma(\tau^- \rightarrow \ell \bar{\nu}_\ell \nu_\tau)$, gives important information about the QCD running coupling at relatively small energy scales. The theoretical analysis of the hadronic decay of a heavy lepton was performed in [1] before the experimental discovery of the τ -lepton in 1975. Since then, the properties of the τ have been studied very intensively. Numerous publications are devoted to the QCD description of the inclusive decay of the τ -lepton and determination of the QCD running coupling α_s at the τ mass scale. A detailed consideration of this subject has been given in [2]. Recently, an updated QCD analysis has been performed by the ALEPH [3] and OPAL [4] collaborations, where applications of different theoretical approaches to the τ -decay have been analyzed.

At present, the R_τ -ratio is known experimentally to high accuracy, $\sim 0.5\%$. Nevertheless, the value of α_s extracted from the data has a rather large error, in which theoretical uncertainties are dominant. For example, the ALEPH Collaboration result is $\alpha_s(M_\tau = 1.777 \text{ GeV}) = 0.334 \pm 0.007_{\text{expt}} \pm 0.021_{\text{theor}}$ [3]. It should be emphasized that nonperturbative terms, the values of which are not well known, do not dominate these uncertainties, because their contribution is rather small [2–4]. The main difficulty is associated with the perturbative description.

The original theoretical expression for the width $\Gamma(\tau^- \rightarrow \text{hadrons } \nu_\tau)$ involves integration over small values of timelike momentum [1]. The perturbative description with the standard running coupling, which has unphysical singularities, becomes ill-defined in this region

and some additional ansatz has to be applied to get a finite result for the hadronic width. To this end, one usually transforms to a contour representation for R_τ [5], which allows one to give meaning to the initial expression and, in principle, perform calculations in the framework of perturbative QCD. Assuming the validity of this transformation it is possible to present results in the form of a truncated power series with $\alpha_s(M_\tau)$ as the expansion parameter [6, 2]. There are also other approaches to evaluating the contour integral. The Le Diberder and Pich prescription [7] allows one to improve the convergence properties of the approximate series and reduce the renormalization scheme (RS) dependence of theoretical predictions. The possibility of using different approaches in the perturbative description of τ -decay leads to an uncertainty in the value of $\alpha_s(M_\tau)$ extracted from the same experimental data. Moreover, any perturbative description is based on this contour representation, *i.e.*, on the possibility of converting the initial expression involving integration over timelike momenta into a contour integral in the complex momentum plane. To carry out this transition by using Cauchy's theorem requires certain analytic properties of the hadronic correlator or of the corresponding Adler function. However, the required analytic properties are not automatically maintained in perturbative QCD resummed by the renormalization group. It is well known that at the one-loop level the so-called ghost pole occurs in the invariant charge. Higher-loop corrections do not solve this problem, but merely add some unphysical branch points. The occurrence of incorrect analytic properties in the conventional perturbative approximation makes it impossible to exploit Cauchy's theorem in this manner and therefore prevents rewriting the initial expression for R_τ in the form of a contour integral in the complex momentum-plane.

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In this paper we will use the analytic approach proposed in [8] (see also [9] for details). Being inspired by Källén–Lehmann analyticity, which is based on general principles of quantum field theory, this method ensures that the running coupling possesses the correct analytic properties, leads to a self-consistent definition of the effective charge in the timelike region [10, 11] (which cannot be a symmetrical reflection of the spacelike one [12]), and provides equality between the initial R_τ -expression and the corresponding contour representation [13]. A distinguishing feature of the analytic approach is the existence of a universal infrared limiting value of the analytic running coupling at $q^2 = 0$ which is independent of both the QCD scale parameter Λ and the choice of renormalization scheme. This limiting value is defined by the general structure of the Lagrangian and turns out to be stable with respect to higher-loop corrections in contrast to the corresponding quantity in conventional perturbation theory (PT). The higher-loop stability of the analytic perturbation theory (APT) holds also for physical observables [13–16].

However, it is not sufficient to study the stability with respect to higher-loop corrections; one must also investigate the stability with respect to choice of renormalization scheme. This is also essential in order to estimate the uncertainty of the results obtained. The theoretical ambiguity which is connected with higher-loop corrections and with RS dependence becomes considerable at low energy scales (see, e.g., [17]). The APT method, as an invariant analytical version of perturbative QCD [18], improves the situation and gives very stable results over a wide range of renormalization schemes. This has been demonstrated for the e^+e^- annihilation ratio [14] and for the Bjorken [15] and Gross–Llewellyn Smith [16] deep inelastic scattering sum rules.

The main aim of the paper is a study of the RS dependence which appears in the description of the inclusive τ decay within the APT approach. We will consider the R_τ -ratio at the next-to-next-to-leading order (NNLO) and the next-to-leading order (NLO) and compare results obtained with those of standard perturbation theory.

2 QCD parametrization of R_τ

The ratio of hadronic to leptonic τ -decay widths can be written as

$$R_\tau = 3 S_{EW} (|V_{ud}|^2 + |V_{us}|^2) (1 + \delta_{QCD}), \quad (1)$$

where $S_{EW} = 1.0194 \pm 0.0040$ [19] is the electroweak factor, $|V_{ud}| = 0.9752 \pm 0.0007$ and $|V_{us}| = 0.2218 \pm 0.0016$ [20] are the CKM matrix elements, and δ_{QCD} is the QCD correction (see [2] for details).

We first introduce some definitions: $\text{Im } \Pi \sim 1 + r$ for the hadronic correlator $\Pi(q^2)$ and $D \sim 1 + d$ for the Adler function $D(q^2)$. Then for massless quarks one can write δ_{QCD} as an integral over timelike momentum s :

$$\delta_{QCD} = 2 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + 2 \frac{s}{M_\tau^2}\right) r(s). \quad (2)$$

Within the conventional perturbative approximation of $r(s)$ this integral is ill-defined due to unphysical singularities of the running coupling lying in the range of integration. The most useful trick to rescue the situation is to appeal to analytic properties of the hadronic correlator $\Pi(q^2)$. This opens up the possibility of exploiting Cauchy's theorem by rewriting the integral in the form of a contour integral in the complex q^2 -plane with the contour being a circle of radius M_τ^2 :

$$\delta_{QCD} = \frac{1}{2\pi i} \oint_{|z|=M_\tau^2} \frac{dz}{z} \left(1 - \frac{z}{M_\tau^2}\right)^3 \left(1 + \frac{z}{M_\tau^2}\right) d(z). \quad (3)$$

Starting from the contour representation (3) the PT description can be developed in the following two ways (see, e.g., [21]). One is Braaten's approach [6] in which the quantity (3) is represented in the form of truncated power series with the expansion parameter $\alpha_s(M_\tau^2)$. The NNLO representation for δ_{QCD} is written as follows

$$\delta_{QCD}^{\text{Br}} = a_\tau + r_1 a_\tau^2 + r_2 a_\tau^3, \quad (4)$$

where $a_\tau \equiv \alpha_s(M_\tau^2)/\pi$. The coefficients r_1 and r_2 in the $\overline{\text{MS}}$ scheme with three active flavors are $r_1 = 5.2023$ and $r_2 = 26.366$ [2, 22]. The running coupling satisfies the renormalization group equation:

$$\mu^2 \frac{da}{d\mu^2} = -\frac{b}{2} a^2 (1 + c_1 a + c_2 a^2), \quad (5)$$

where b , c_1 and c_2 are the β -function coefficients. For three active flavors $b = 9/2$, $c_1 = 16/9$ and $c_2^{\overline{\text{MS}}} = 3863/864$.

In the approach of Le Diberder and Pich (LP) [7], the PT expansion is applied to the d -function¹

$$d(q^2) = a(q^2) + d_1 a^2(q^2) + d_2 a^3(q^2), \quad (6)$$

where in the $\overline{\text{MS}}$ -scheme $d_1^{\overline{\text{MS}}} = 1.6398$ and $d_2^{\overline{\text{MS}}} = 6.3710$ [22] for three active quarks. Substituting (6) into (3) leads to the following expansion, which is not a power series in a ,

$$\delta_{QCD}^{\text{LP}} = A^{(1)}(a) + d_1 A^{(2)}(a) + d_2 A^{(3)}(a) \quad (7)$$

with

$$A^{(n)}(a) = \frac{1}{2\pi i} \oint_{|z|=M_\tau^2} \frac{dz}{z} \left(1 - \frac{z}{M_\tau^2}\right)^3 \left(1 + \frac{z}{M_\tau^2}\right) a^n(z). \quad (8)$$

As noted above, transition to the contour representation (3) requires certain analytic properties of the correlator, namely, that it must be an analytic function in the complex q^2 -plane with a cut along the positive real axis. The correlator parametrized by the PT running coupling does not have this virtue [23, 13]. Moreover, the conventional renormalization group method determines the running coupling in the spacelike region, whereas the initial

¹ We use the definition $q^2 < 0$ in the Euclidean region. We have made a few changes in notation from that given in [13]: now $a = \alpha_s/\pi$, and consequently d_1 and d_2 are what we called d_2 and d_3 previously.

expression (2) for R_τ contains an integration over time-like momentum. Thus, we are in need of some method of continuing the running coupling from the spacelike to the timelike region that takes into account the proper analytic properties of the running coupling [11]. Because of this failure of analyticity, (2) and (3) are not equivalent in the framework of PT [13] and if one remains within PT, nothing can be said about the errors introduced by this transition.

The analytic approach may eliminate these problems. To make our analysis more transparent and to demonstrate clearly the differences between the consequences of the PT and APT methods, we restrict our consideration here to massless NNLO. The NNLO analysis can be performed in a more rigorous way without model assumptions that allows us to avoid minor details and exhibit the principal features of the APT approach. Thus, other effects, such as nonperturbative terms, higher-loop corrections, and renormalon contributions lie outside of the purpose of this paper. Note, however, that the NNLO approximation is adequate to describe the actual physical situation because numerically the corresponding terms give the principal contribution to the R_τ -ratio.

The function $d(q^2)$, which is analytic in the cut q^2 -plane, can be expressed in terms of the effective spectral function $\rho(\sigma)$, the basic quantity in the APT method,

$$d(q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma - q^2} \rho(\sigma). \quad (9)$$

The connection between the QCD corrections to the D - and R -functions can be written down in the form of the dispersion integral

$$d(q^2) = -q^2 \int_0^\infty \frac{ds}{(s - q^2)^2} r(s), \quad (10)$$

which is inverted by the following formula [24]

$$r(s) = -\frac{1}{2\pi i} \int_{s-i\epsilon}^{s+i\epsilon} \frac{dz}{z} d(z). \quad (11)$$

Here, the contour lies in the region of analyticity of the D -function. In terms of $\rho(\sigma)$ the function $r(s)$ defined for timelike momenta can be expressed as follows [10]:

$$r(s) = \frac{1}{\pi} \int_s^\infty \frac{d\sigma}{\sigma} \rho(\sigma). \quad (12)$$

(9) and (12) determine the QCD corrections $d(q^2)$, which is defined in the Euclidean (spacelike) region of momenta, and $r(s)$ defined for the Minkowskian (timelike) argument, in terms of the spectral function $\rho(\sigma)$. For δ_{QCD} , using (2) or equivalently (3), in terms of $\rho(\sigma)$, we find²

$$\delta_{\text{an}} = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma} \rho(\sigma) - \frac{1}{\pi} \int_0^{M_\tau^2} \frac{d\sigma}{\sigma} \times \left(1 - \frac{\sigma}{M_\tau^2}\right)^3 \left(1 + \frac{\sigma}{M_\tau^2}\right) \rho(\sigma). \quad (13)$$

² To distinguish APT and PT cases, we will use subscripts “an” and “pt”.

Table 1. Successive loop contributions to the PT and APT expansions for $R_\tau/[3 S_{\text{EW}}(|V_{ud}|^2 + |V_{us}|^2)]$

Method of description	Expansion terms
PT (Br)[6]	$1 + \delta_{\text{pt}}^{\text{Br}} = 1 + 0.104 + 0.056 + 0.030$
PT (LP)[7]	$1 + \delta_{\text{pt}}^{\text{LP}} = 1 + 0.148 + 0.030 + 0.012$
APT [8]	$1 + \delta_{\text{an}} = 1 + 0.167 + 0.021 + 0.002$

In the APT approach, the spectral function is defined as the imaginary part of the perturbative approximation to $d_{\text{pt}}(q^2)$ on the physical cut:

$$\rho(\sigma) = \varrho_0(\sigma) + d_1 \varrho_1(\sigma) + d_2 \varrho_2(\sigma), \quad (14)$$

where

$$\varrho_n(\sigma) = \text{Im}[a^{n+1}(\sigma + i\epsilon)]. \quad (15)$$

Substituting (14) into (13), we can rewrite δ_{an} in the form of the APT expansion

$$\delta_{\text{an}} = \delta^{(0)} + d_1 \delta^{(1)} + d_2 \delta^{(2)}. \quad (16)$$

Note that the APT representations of the d -function and the QCD correction δ_{QCD} are not in the form of power series.

The function $\varrho_0(\sigma)$ in (14) defines the analytic spacelike running coupling

$$a_{\text{an}}(q^2) = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma - q^2} \varrho_0(\sigma). \quad (17)$$

In the one-loop approximation it leads to [8]

$$a_{\text{an}}(q^2) = a_{\text{pt}}(q^2) + \frac{2}{b} \frac{\Lambda^2}{\Lambda^2 + q^2}. \quad (18)$$

Unlike the one-loop PT running coupling, $a_{\text{pt}}(q^2) = 2/b \times \ln(-q^2/\Lambda^2)$, the analytic running coupling (18) has no unphysical ghost pole and, therefore, possesses the correct analytic properties, arising from Källén-Lehmann analyticity that reflects the general principles of the theory. The nonperturbative (non-logarithmic) term, which appears in the analytic running coupling, does not change the ultraviolet limit of the theory and thus the APT and the PT approaches coincide with each other in the asymptotic region of high energies.

Thus, the APT approach provides a self-consistent description of the hadronic τ decay. This description can be equivalently phrased either on the basis of the original expression (2), which involves the Minkowskian quantity $r(s)$, or on the contour representation (3), which involves the Euclidean quantity $d(q^2)$.

An important feature of the APT approach is the fact that $d_{\text{an}}(q^2)$ and $a_{\text{an}}(q^2)$ have a universal limit at the point $q^2 = 0$. This limiting value, generally, is independent of both the scale parameter Λ and the order of the loop expansion being considered. Because $d_{\text{an}}(0)$ and $a_{\text{an}}(0)$ are equal to the reciprocal of the first coefficient of the QCD

Table 2. QCD parameters extracted from $R_\tau = 3.642 \pm 0.019$ [25] in the $\overline{\text{MS}}$ scheme

Approximation	Method	$\Lambda_{\overline{\text{MS}}} \text{ (MeV)}$	$\alpha(M_\tau^2)$
NNLO	PT (Br)	366 ± 14	0.328 ± 0.007
	PT (LP)	391 ± 16	0.340 ± 0.008
	APT	907 ± 94	0.403 ± 0.015
NLO	PT (Br)	492 ± 17	0.371 ± 0.009
	PT (LP)	465 ± 19	0.358 ± 0.009
	APT	954 ± 90	0.404 ± 0.014

β -function, they are also RS invariant (we consider only gauge- and mass-independent RSs). The existence of this fixed point plays a decisive role in the improved convergence properties relative to PT and in the very weak RS dependence of our results.

To find the analytic function $d(q^2)$ involved in (3), we solve the transcendental equation for the running coupling

$$\frac{b}{2} \ln \left(\frac{-q^2}{\Lambda_{\overline{\text{MS}}}^2} \right) - i\pi \frac{b}{2} = d_1^{\overline{\text{MS}}} - d_1 + \frac{1}{a} + c_1 \ln \left(\frac{b}{2c_1} \right) + F^{(l)}(a), \quad (19)$$

where at the NLO

$$F^{(2)}(a) = c_1 \ln \left(\frac{c_1 a}{1 + c_1 a} \right), \quad (20)$$

and at the NNLO

$$F^{(3)}(a) = F^{(2)}(a) + c_2 \int_0^a \frac{dx}{(1 + c_1 x)(1 + c_1 x + c_2 x^2)}, \quad (21)$$

on the physical cut lying along the positive real axis in the complex q^2 -plane and then use (14), (15) and (9). (19) holds in any MS-like renormalization scheme and allows us to normalize the results obtained by using the scale parameter $\Lambda_{\overline{\text{MS}}}$. Having found $\Lambda_{\overline{\text{MS}}}$, we can study how δ_{an} varies with a change of renormalization scheme. To do that one has to select parameters which determine the RS. The function $d(q^2)$ in (3) is parametrized by a set of RS-dependent parameters. There are RS invariant combinations which constrain these parameters [27]. At the NNLO there are two RS-invariant quantities; the first of them expresses an energy dependence, the second is just a number

$$\omega_2 = c_2 + d_2 - c_1 d_1 - d_1^2, \quad (22)$$

which in our case equals 5.2378. Here, c_1 is RS invariant and we can choose d_1 and c_2 as independent variables, which define some RS.

There are no fundamental principles upon which one can choose one or another preferable RS. Nevertheless, a natural way of studying the RS dependence is to supplement results in a certain scheme with an estimate of the variability of the predictions over a range of *a priori* acceptable schemes specified by some criterion. In [28] it

Table 3. NLO and NNLO predictions for δ_{an} in the $\overline{\text{MS}}$ scheme

M_τ^2/Λ^2	$\delta_{\text{an}}^{\text{NLO}}$	$\delta_{\text{an}}^{\text{NNLO}}$	M_τ^2/Λ^2	$\delta_{\text{an}}^{\text{NLO}}$	$\delta_{\text{an}}^{\text{NNLO}}$
2.0	0.2090	0.2106	4.5	0.1820	0.1857
2.5	0.2016	0.2039	5.0	0.1785	0.1824
3.0	0.1955	0.1983	5.5	0.1753	0.1795
3.5	0.1904	0.1935	6.0	0.1724	0.1767
4.0	0.1859	0.1894	6.5	0.1698	0.1743

was proposed to consider the class of ‘natural’ RSs, which obey the condition

$$|c_2| + |d_2| + c_1 |d_1| + d_1^2 \leq C |\omega_2|. \quad (23)$$

This inequality is called the ‘cancellation index criterion’ which means that the degree of cancellation in the second RS invariant (22) should not be too large. To define a boundary of ‘acceptable’ schemes which is defined by the value of the cancellation index C , we will require no more cancellation than that which occurs in the scheme obeying the principle of minimal sensitivity (PMS) [29], which leads to $C \simeq 2$.

3 APT: convergence properties and RS dependence

For various physical quantities, the APT approach allows one to construct a series that has improved convergence properties as compared to a perturbative expansion. To demonstrate this fact for the hadronic τ -decay, we compare the convergence properties of the PT expansions (4) and (7) on the one hand, and the APT approach given by (16) on the other hand. For our calculation we take as input the TAU’98 conference value: $R_\tau = 3.642 \pm 0.019$ [25], which is consistent with the PDG’98 fit $R_\tau = 3.642 \pm 0.024$ [20]. In Table 1 we present NNLO results obtained by the methods mentioned above for the central experimental value in the $\overline{\text{MS}}$ scheme. The relative contributions of higher-order terms depends on the method which is applied. The convergence properties of the APT expansion seem to be much improved compared to those of the PT expansions.

The values of the scale parameter $\Lambda_{\overline{\text{MS}}}$ and the coupling $\alpha_s(M_\tau^2)$ obtained from above PT and APT expansions are noticeably different from each other. The corresponding numerical estimations are given in Table 2, in which, in order to clarify the situation concerning higher-loop stability of different expansions, we also present the NLO result. This table demonstrates that the theoretical ambiguity, which is associated with different versions of the perturbative description, leads to a rather large uncertainty, $\alpha_{\text{PT (Br)}}^{\text{NNLO}} - \alpha_{\text{PT (LP)}}^{\text{NNLO}} = 0.012$. At the same time the experimental error is $\Delta\alpha_{\text{expt}} = 0.007\text{--}0.009$ [3,4]. The distinction between NLO and NNLO running coupling values is 12% for PT (Br) and 5% for PT (LP) approaches, while for the APT approach it is less than 0.5%.

The non-logarithmic terms, which ensure the correct analytic properties and allow a self-consistent description of τ decay, turn out to be very important for the numerical analysis and influence in an essential way the value of the Λ parameter extracted from the data. Indeed, at the one-loop level one can write a simple relation: $\delta_{\text{an}}(\Lambda) \simeq \delta_{\text{pt}}^{\text{LP}}(\Lambda) - (2/b)\Lambda^2/M_\tau^2$. The second term, which is ‘invisible’ in the perturbative expansion, turns out to be numerically important [26] (see the detailed discussion in [11]). Note also that there is a difference between the shapes of the analytic and perturbative running couplings, for example, $\alpha_{\text{an}}(\Lambda = 907 \text{ MeV}) = 0.403$, while at the same scale, the value of the perturbative coupling much larger, $\alpha_{\text{pt}}(\Lambda = 907 \text{ MeV}) = 0.796$. Here the question may arise, how is the large APT value of Λ consistent with high energy experimental data? We have estimated the ratio of hadronic to leptonic Z-decay widths, R_Z , using the above value of Λ_{an} and the matching procedure proposed in [11]. We obtained the value $R_Z = 20.82$, which lies within the range of experimental errors; for example, the PDG’98 average is $R_Z = 20.77 \pm 0.07$ [20]. This fact can be understood if one takes into account that there are differences between the shapes of the analytic and perturbative running couplings and also in the terms of the corresponding series.

We found that the value of δ_{an} depends so slightly on $A_{\overline{\text{MS}}}$ that a 0.9% error in R_τ gives 18% error in the value of $A_{\overline{\text{MS}}}$. (This is the reason why the errors in the values of $A_{\overline{\text{MS}}}$ and $\alpha(M_\tau^2)$ given by APT are larger than those in PT.) We illustrate this feature in Table 3. According to the table, when we change M_τ^2/Λ^2 from 2.0 to 6.5 (corresponding to a variation of Λ from 1.256 GeV to 0.697 GeV), δ_{an} is only altered by about 20%. The sensitivity to $A_{\overline{\text{MS}}}$ increases as M_τ^2/Λ^2 gets smaller.

Consider now the RS dependence of the APT result and compare it with the perturbative LP approach,³ which of the two PT schemes is more preferable from the viewpoint of sensitivity to RS dependence. In the framework of the PT, the RS dependence of δ_{QCD} has been discussed in detail in [29].

In the $\overline{\text{MS}}$ scheme we adopt $\delta_{\text{an,pt}} = 0.1906$ and consider some RS belonging to the domain described above [see (23)]. Take two schemes, A and B , located at the two lower corners of the boundary of the domain (see Fig. 1), *i.e.*, they have the same cancellation index as does the $\overline{\text{MS}}$ scheme, with $A = (-1.6183, 0)$ and $B = (0.9575, 0)$, where the first coordinate is d_1 and the second is c_2 . Then for the PT case in NNLO we get $\delta_{\text{pt}}(A) = 0.2025$ and $\delta_{\text{pt}}(B) = 0.1911$. Therefore, even for this sufficiently narrow class of RS the perturbative approach gave a 6% deviation in δ_{QCD} that corresponds to a RS uncertainty for the running coupling value in the $\overline{\text{MS}}$ -scheme of $\Delta\alpha_{\text{pt}}^{\text{RS}} = 0.0153$. The difference between APT results is much smaller: $\delta_{\text{an}}(A) = 0.1890$ and $\delta_{\text{an}}(B) = 0.1905$, and we have only 0.8% deviation, which corresponds in the $\overline{\text{MS}}$ -scheme to a RS uncertainty of $\Delta\alpha_{\text{an}}^{\text{RS}} = 0.0035$.

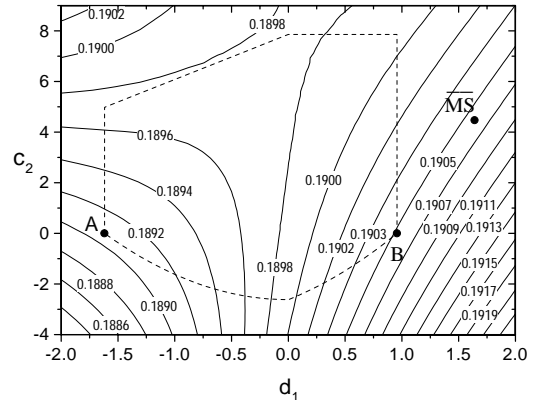


Fig. 1. Contour plot of values of δ_{an} at the three-loop order as a function of RS parameters d_1 and c_2 . The dashed line indicates the boundary of the domain, defined by (23) with $C = 2$, the heavy points are the positions of the A , B and $\overline{\text{MS}}$ schemes

The similar RS stability holds also at the two-loop level: one has a 5% deviation in the PT case and only a 0.4% for the APT. We display our NNLO results in the form of a contour plot, in Fig. 1.

It is worthwhile to analyze some schemes lying outside the domain considered above with the relatively small value of the cancellation index $C \simeq 2$. Among them there is, for instance, the commonly used $\overline{\text{MS}}$ scheme which does not belong this domain. In [29] it was shown that the so-called V scheme [30] lies very far from the domain described above and gives so a large value of δ_{pt} that it cannot be used at this low energy. For the V scheme we have $d_1 = -0.109$ and $c_2 = 26.200$. The three-loop perturbative result is $\delta_{\text{pt}}(V) = 0.3060$ that corresponds to about a 61% deviation from the $\overline{\text{MS}}$ scheme. On the other hand, if we turn to APT we have $\delta_{\text{an}} = 0.1902$, *i.e.*, only about a 0.2% deviation from the $\overline{\text{MS}}$ scheme. So the V scheme is still useful at this energy in APT.

In PT at high energies the weak RS dependence is a consequence of the small value of the coupling constant. At lower energies the uncertainty increases. In APT, at high energies, the situation is the same. However, at low energies the theory has a universal RS-invariant infrared limiting value $d_{\text{an}}(0)$, which restricts the RS ambiguity over a very wide range of momentum. Another way to illustrate the remarkable stability of APT is to calculate the spectral functions $\varrho_n(\sigma)$ given by (15); one sees that $\varrho_1(\sigma)$ is much smaller than $\varrho_0(\sigma)$ over the whole spectral region. The same statement is true for the relationship between $\varrho_1(\sigma)$ and $\varrho_2(\sigma)$. This monotonically decreasing behavior reduces the RS dependence strongly, since the perturbative coefficients d_1 and d_2 in expression (14) for $\rho(\sigma)$ are multiplied by these functions. For the $\overline{\text{MS}}$ scheme, this situation is demonstrated in Fig. 2.

³ The LP approach is often called the contour-improved fixed-order PT (CIPT) [25].

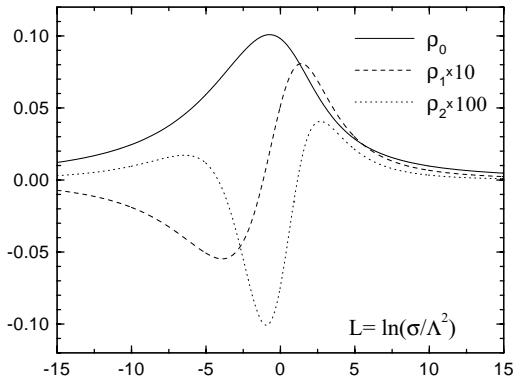


Fig. 2. The spectral densities ρ_0 , $\rho_1 \cdot 10$ and $\rho_2 \cdot 10^2$ vs. $L = \ln(\sigma/\Lambda^2)$ in the $\overline{\text{MS}}$ scheme

4 Conclusion

We have considered inclusive τ -decay in three-loop order within analytic perturbation theory concentrating on the analysis of theoretical uncertainties coming from the perturbative short distance part of the QCD correction to the R_τ -ratio, which defines the principal contribution to this physical quantity. For the low energy τ -mass scale, the main source of theoretical uncertainties results from the inevitable truncation of the perturbative series, which leads to the essential RS dependence and higher loop sensitivity of the theoretical predictions. In order to resolve this problem within the conventional perturbative approach it is possible to try, in principle, to compute higher loop contributions. However, even if this were to be done, one has to keep in mind that from the rigorous point of view it will hardly be sufficient because the series is asymptotic, and, in any finite order, the analytic properties of the hadronic correlator, which arise from general principles of the theory, are violated. Thus, to resolve this problem one has to use a modification of the perturbative expansion at low energy scales.

Here, we have applied the analytic approach which is not inconsistent with the general principles of quantum field theory and which opens up the possibility of reducing the theoretical uncertainties associated with short distance contributions mentioned above. Let us summarize the important features of this method: (i) the method maintains the correct analytic properties and leads to a self-consistent definition of the procedure of analytic continuation from the spacelike to the timelike region; (ii) the APT approach has much improved convergence properties and turns out to be stable with respect to higher-loop corrections; (iii) the RS dependence of the results obtained is reduced drastically. For example, the V scheme, which gives a very large discrepancy in standard perturbation theory, can be used in analytic perturbation theory without any difficulty and the APT predictions are practically RS independent over a wide region of RS parameters.

The nonperturbative power corrections coming from the operator product expansion (in this connection see a discussion in [31,32]), renormalon and other effects are beyond the scope of our present consideration. Note, however, that the process of enforcing analyticity modifies the perturbative contributions by incorporating some nonperturbative terms. The form of the APT running coupling and the non-power structure of the APT expansion are essentially different from the PT ones. Numerically, this difference becomes very important in the region less than a few GeV and in order to get the same physical quantity the contribution of power corrections should also be changed.

The value of Λ_{APT} is very sensitive to the experimental value of R_τ . For example, as has been demonstrated in [13] the use of the value of R_τ obtained by the CLEO collaboration [33] gives a value of the scale parameter some 30% smaller than that found here. Note also that the renormalon contribution influences the value of Λ extracted from the τ data (see [34] for a review). Within the usual approach, renormalons reduce the value of $\alpha_s(M_\tau^2)$ by about 15%. A similar situation holds also in APT and for the nonperturbative a -expansion approach [35], which allows one, as in APT, to maintain the required analyticity [23]. These two analytic approaches often lead to rather similar consequences. For example, they allow one to get a good description of experimental data corresponding to the Euclidean and Minkowskian characteristics of the process of e^+e^- annihilation into hadrons down to the lowest energy scale [14,36].

Pure massless APT analysis, which has been performed here, leads to an unusually large value of the QCD scale parameter Λ as compared to the conventional PT value. This is connected with the presence of nonperturbative contributions that appear in the APT method which have a negative relative sign. The effects mentioned above can change the value of the scale parameter extracted from the τ data. However, this fact is not relevant for the essential conclusion which we have claimed in this paper, that the APT method provides predictions which are stable with respect to the choice of renormalization scheme and to the inclusion of higher loop corrections. Thus, the analytic approach discussed here is not in conflict with the general principles of the theory and allows one to reduce the uncertainties of theoretical predictions drastically.

Acknowledgements. The authors would like to thank D.V. Shirkov for interest in this work. Partial support of the work by the US National Science Foundation, grant PHY-9600421, and by the US Department of Energy, grant DE-FG-03-98ER41066, and by the RFBR, grants 99-01-00091, 99-02-17727 is gratefully acknowledged. The work of ILS and OPS is also supported in part by the University of Oklahoma, through its College of Arts and Science, the Vice President for Research, and the Department of Physics and Astronomy.

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